Math 1540 Final - Extra Study Guide

1. Be able to use the product rule: If $y = f(x)g(x)$, then $y' = f(x)g'(x) + g(x)f'(x)$

$$z = 8x^2e^x \qquad 8x^2e^x + \underline{e^x\,16x}$$

$$\frac{dz}{dx} = 16xe^x + 8x^2e^x \qquad = \quad 16xe^x + 8x^2e^x$$

$$-5x^{1/2}$$

$$f(x) = (2x^5 - 2x^3 - 6)(5x^2 - 5\sqrt{x}) \quad = \quad \underline{(2x^5-2x^3-6)}\left(10x - \frac{5}{2}x^{-1/2}\right) + (5x^2-5\sqrt{x})(10x^4-6x^2)$$

$$f'(x) = (2x^5 - 2x^3 - 6)\left[10x - \frac{5}{2\sqrt{x}}\right] + (5x^2 - 5\sqrt{x})(10x^4 - 6x^2)$$

2. Know how to use the quotient rule: If $y = \frac{f(x)}{g(x)}$ Then $y' = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$

$$y = \frac{5x^2 + x - 1}{x^3 - 9x^2}$$

$$y' = \frac{(x^3-9x^2)(10x+1) - (5x^2+x-1)(3x^2-18x)}{(x^3-9x^2)^2}$$

$$= \frac{10x^4 + x^3 - \cancel{90x^3} - 9x^2 - 15x^4 + \cancel{90x^3} - 3x^3 + 18x^2 + 3x^2 - 18x}{(x^3-9x^2)^2}$$

$$\frac{dy}{dx} = \frac{-5x^4 - 2x^3 + 12x^2 - 18x}{(x^3 - 9x^2)^2}$$

$$= \frac{-5x^4 - 2x^3 + 12x^2 - 18x}{(x^3-9x^2)^2}$$

$$y = \frac{x^2 + 2x - 2}{x^2 - 2x + 2}$$

$$\frac{dy}{dx} = \frac{-4x^2 + 8x}{(x^2 - 2x + 2)^2}$$

$$\frac{(x^2-2x+2)(2x+2) - (x^2+2x-2)(2x-2)}{(x^2-2x+2)^2}$$

$$= \frac{2x^3 + 2x^2 - 2x^2 - 4x + 4x + 4 - 2x^3 + 2x^2 - 4x^2 + 4x + 4x - 4}{(x^2-2x+2)^2}$$

$$= \frac{-4x^2 + 8x}{(x^2-2x+2)^2}$$

②

3. Know how to use the Chain Rule (see Chain Rule video on web site)

If $y = f(u), and\ u = g(x),\ then\ y' = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$$g(x) = \left(5x^4 + 4x + \frac{4}{x^2}\right)^{8/5} = \frac{8}{5}\left(5x^4 + 4x + \frac{4}{x^2}\right)^{3/5}\left(20x^3 + 4 - 8x^{-3}\right)$$

$$g'(x) = \frac{8}{5}\left(5x^4 + 4x + \frac{4}{x^2}\right)^{3/5}\left(20x^3 + 4 - \frac{8}{x^3}\right)$$

ok

$$f(x) = (2x^5 - 4x^4 + 3)^{308}$$

$$f'(x) = 308(2x^5 - 4x^4 + 3)^{307}(10x^4 - 16x^3)$$

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4. Know how to calculate integrals using the substitution rule (see video on web site)

$$\int_0^1 \underbrace{\frac{4x^3}{(1+x^4)^4}\,dx}_{du} = \frac{7}{24}$$

$$\int \frac{t^3}{\sqrt[4]{3+t^4}}\,dt = \frac{1}{3}(3+t^4)^{3/4} + C$$

$\frac{du}{4}$

$u = 1 + x^4, \quad \frac{du}{dx} = 4x^3$

$du = 4x^3\,dx$

$u_{upper} = 2, \quad u_{lower} = 1$

$$\int_1^2 \frac{du}{u^4} = \int_1^2 u^{-4}\,du = \frac{-1}{3}u^{-3}\Big|_1^2 = \frac{-1}{3u^3}\Big|_1^2$$

$$= \frac{-1}{24} + \frac{1}{3} = \frac{-1}{24} + \frac{8}{24} = \frac{7}{24}$$

$u = 3 + t^4$

$\frac{du}{dt} = 4t^3, \quad du = 4t^3\,dt$

$t^3\,dt = \frac{du}{4}$

$$\int \frac{du}{4(u^{1/4})}\,du$$

$$= \int \frac{1}{4} u^{-1/4}\,du = \frac{1}{4}\frac{u^{3/4}}{3/4} + C$$

$$= \frac{u^{3/4}}{3} + C$$

$$= \frac{(3+t^4)^{3/4}}{3} + C$$

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5. Understand implicit differentiation

Suppose that x and y are related by the equation $x^3 + (2y + 1)^2 = y^2$. Use implicit differentiation to determine $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{-3x^2}{2(3y + 2)}$$

$$\frac{d}{\partial x} x^3 + \frac{d}{\partial x}(2y+1)^2 = \frac{d}{\partial x} y^2$$

$$3x^2 + 2(2y+1)\left(2\frac{dy}{dx}\right) = 2y\frac{dy}{\partial x}$$

$$3x^2 + 8y\frac{dy}{\partial x} + 4\frac{dy}{\partial y} = 2y\frac{dy}{\partial x}$$

$$6y\frac{dy}{\partial x} + 4\frac{dy}{x} = -3x^2$$

$$\frac{dy}{\partial x}(6y+4) = -3x^2$$

$$\frac{dy}{\partial x} = \frac{-3x^2}{2(3y+2)}$$

Use implicit differentiation to find dy/dx.

$\cos xy + x^7 = y^7$

dy/dx $= \frac{7x^6 - y \sin xy}{7y^6 + x \sin xy}$

$$\frac{d}{\partial x} \cos xy + \frac{d}{\partial x} x^7 = \frac{d}{\partial x} y^7$$

$$-\sin xy \frac{d}{\partial x}(xy) + 7x^6 = 7y^6\frac{dy}{dx}$$

$$-\sin xy\left(x\frac{dy}{\partial x} + y\right) + 7x^6 = 7y^6\frac{dy}{\partial x}$$

$$-x \sin xy \frac{dy}{\partial x} - y\sin xy + 7x^6 = 7y^6\frac{dy}{\partial x}$$

$$7y^6\frac{dy}{\partial x} + x\sin xy\frac{dy}{\partial x} = 7x^6 - y\sin xy$$

$$\frac{dy}{\partial x} = \frac{7x^6 - y\sin xy}{7y^6 + x\sin xy}$$

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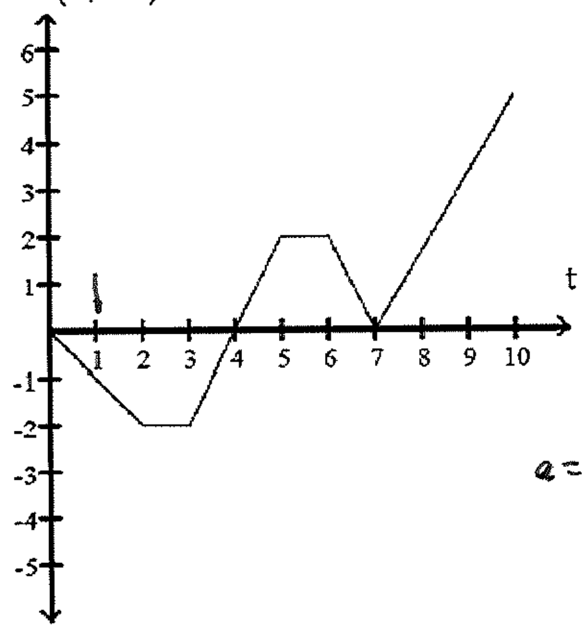
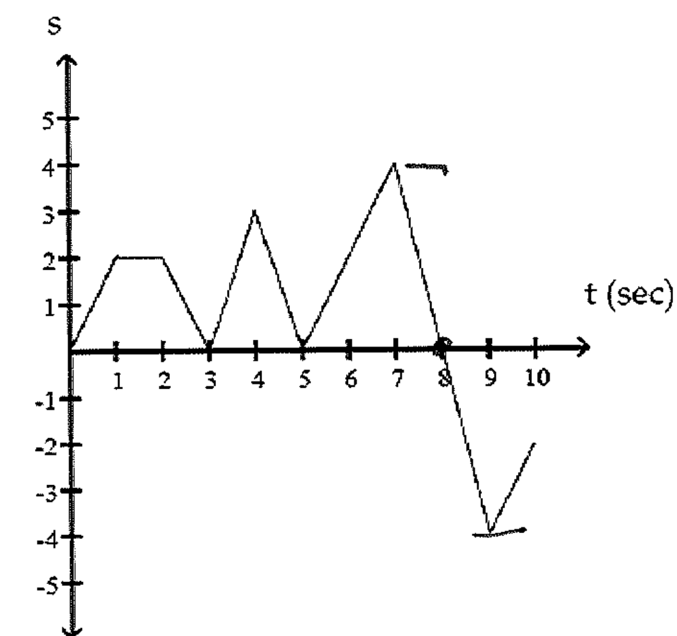
6. Position s, velocity v, and acceleration.

$v = s'$, $a = v' = s''$ To get velocity, you need the derivative of the position. To get acceleration, you need to take the derivative of the velocity or the second derivative of the position.



$a = \frac{dv}{dt} = \frac{-2}{2} \frac{ft/sec}{sec} = -1 \ ft/sec/sec$

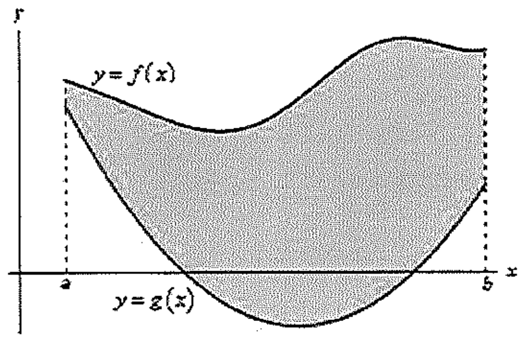
What is the body's acceleration when $t = 1$ sec?



What is the body's velocity when t = 8 sec?

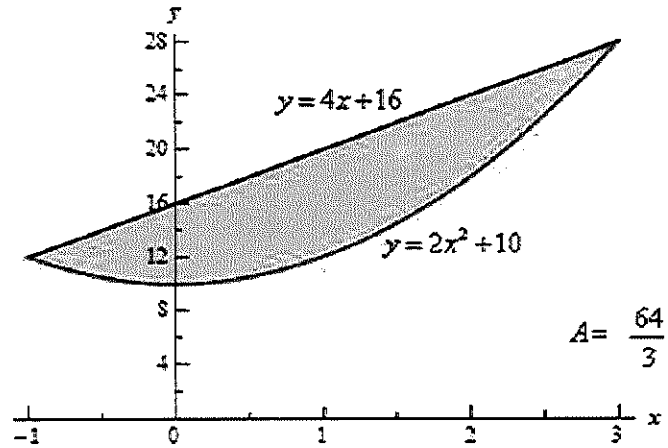
$$V = \frac{ds}{\partial t} = \frac{-4-4}{2} = -\frac{8}{2} = -4 \text{ ft/sec}$$

7. Area between curves



$$A=\int_a^b f(x)-g(x)\,dx$$

$$A=\int_a^b \left(\begin{matrix}\text{upper}\\ \text{function}\end{matrix}\right)-\left(\begin{matrix}\text{lower}\\ \text{function}\end{matrix}\right)dx,$$



$$\int_{-1}^{+3} 4x+16-(2x^2+10)\,dx$$

$$= \int_{-1}^{+3} (4x+16-2x^2-10)\,dx = \int_{-1}^{+3} 4x+6-2x^2\,dx$$

$$= \left(2x^2+6x-\frac{2x^3}{3}\right)\Big|_{-1}^{3}$$

$$= 18+\cancel{18}-\cancel{18}-(2-6+2/3)$$

$$= 18-2+6-2/3 = 22-\frac{2}{3}$$

$$= \frac{66}{3}-2/3 = \frac{64}{3}$$

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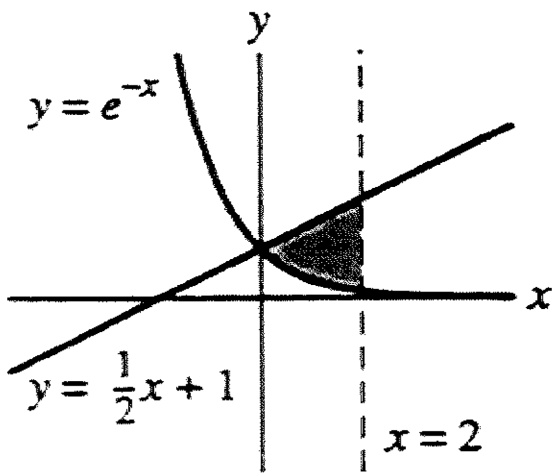
Use the graph below to determine the area of the shaded region.



answer $2 + e^{-2}$

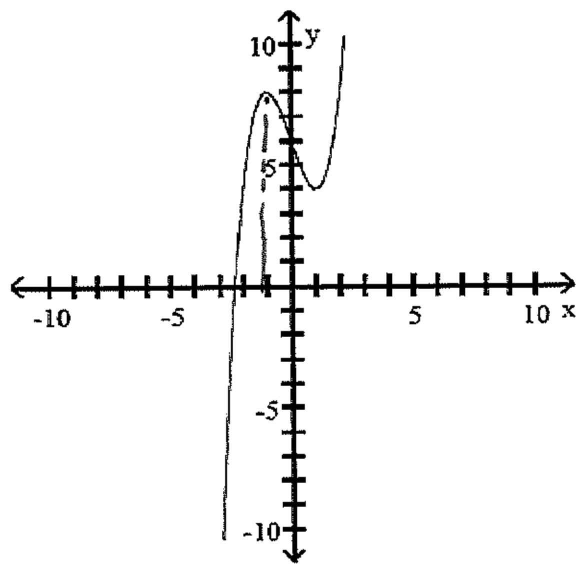
$$\int_0^2 \underbrace{\tfrac{1}{2}x + 1} - e^{-x} \, dx$$

$$= \frac{x^2}{4} + x + e^{-x} \Big|_0^2$$

$$= 1 + 2 + e^{-2} - e^{-0}$$

$$= 3 + e^{-2} - 1 = 2 + e^{-2}$$

9. A function y = f(x) is increasing if y' > 0 and is decreasing if y' < 0. A graph is concave up if y'' > 0 and concave down if y'' < 0 On the graph below, state where y' > 0 and < 0. Label local min and max. Label intervals where function is concave up or down and where it is concave up.



$y' > 0 \quad (-\infty, -1) \quad (1, \infty)$

$y' < 0 \quad (-1, +1)$

Concave up $(0, \infty)$

Down $(-\infty, 0)$

Local max $(-1, 8)$

Local min $(+1, 4)$

10. To find the equation of the tangent line to a graph at a point (x_1, y_1), you must find

$y' = f'(x_1) = m$ which will be the slope of the tangent line at the point.

Then write $y_1 = mx_1 + b$ and solve $b = y_1 - mx_1$ Then the equation is $y = mx + b$

Find an equation of the tangent line at the indicated point on the graph of the function.

1) $y = f(x) = x - x^2$, $(x, y) = (4, -12)$

$y = -7x + 16$

$\longrightarrow$ $y' = 1 - 2x$

$y'\Big|_{x=4} = 1 - 2(4) = -7 = m$

$y = -7x + b \quad (4, -12)$

$-12 = -7(4) + b$

$b = -12 + 28 = 16$

$\boxed{y = -7x + 16}$

2) $y = f(x) = 10\sqrt{x} - x + 1$, $(x, y) = (100, 1)$

$y = -\frac{1}{2}x + 51$

$y = 10x^{1/2} - x + 1$

$y' = 5x^{-1/2} - 1 = \frac{5}{\sqrt{x}} - 1$

$y'(100) = \frac{5}{\sqrt{100}} - 1 = \frac{5}{10} - 1 =$

$= \frac{1}{2} - 1 = -\frac{1}{2} = m$

$y = -\frac{1}{2}x + b \quad (100, 1)$

$1 = -\frac{1}{2}(100) + b$

$b = 1 + 50 = 51$

$\boxed{y = -\frac{1}{2}x + 51}$

11. Finding limits. a) Plug in b) Use calculator, c) factor, d)use "trick" or try L'Hopital's Rule

$$\lim_{x \to -1} \frac{x^2 - 1}{\sqrt{x^2 + 3} - 2} = \frac{4}{}$$

L'Hopital's Rule

$$= \lim_{x \to -1} \frac{\cancel{2x}}{\frac{\cancel{2x}}{2\sqrt{x^2+3}}} = \lim_{x \to -1} 2\sqrt{x^2+3}$$

$$= 2\sqrt{4} = 4$$

$$\lim_{x \to 0} \frac{\sqrt{36 + x} - \sqrt{36 - x}}{x} = \frac{1}{6}$$

$$\lim_{x \to 0} \frac{\frac{1}{2\sqrt{36+x}} + \frac{1}{2\sqrt{36-x}}}{1}$$

$$= \frac{1}{2\sqrt{36}} + \frac{1}{2\sqrt{36}} = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$$

12. Finding maxima and minima: a) Calculate critical points (y' = 0 or undefined) b) Determine concavity at critical points. y'' > 0 (concave up ∪ relative min), y'' < 0 (concave down ∩ relative max) c) Check y value at any given end points if y is defined on an interval [a, b].

$$f(x) = \frac{7x}{x^2+1}$$

Relative minimum: $\left(-1, -\frac{7}{2}\right)$, relative maximum: $\left(1, \frac{7}{2}\right)$

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$$f'(x) = \frac{(x^2+1)7 - 7x(2x)}{(x^2+1)^2} = \frac{7x^2+7-14x^2}{(x^2+1)^2} = \frac{-7x^2+7}{(x^2+1)^2} = 0$$

$-7x^2+7=0$

$7x^2 = 7$

$x^2 = 1$

$x=-1, \quad x=+1$

$$f''(x) = \frac{(x^2+1)^2(-14x) - (-7x^2+7)2(x^2+1)(2x)}{(x^2+1)^2}$$

$$= \frac{(x^2+1)\left[(x^2+1)(-14x) + -4x(7x^2-7)\right]}{(x^2+1)^2}$$

$$= \frac{-14x^3 - 14x + 28x^3 - 28x}{(x^2+1)}$$

$$\frac{14x^3 - 42x}{(x^2+1)} = \frac{14x(x^2-3)}{(x^2+1)}$$

$$\cancel{\frac{-14(x^3+x+2x^2-2)}{(x^2+1)}}$$

$$= \frac{-14(x^3+x+2x^2-2)}{(x^2+1)}$$

$x^2 = 3$

$x = \sqrt{3}$

$x=-1 \quad \frac{-14(-2)}{2} > 0$ concave up

$x=+1 \quad \frac{14(-2)}{2} < 0$ concave down

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$f(x) = -x^3 + 3x^2 - 2$

Relative minimum: (0, −2); relative maximum: (2, 2)

$f'(x) = -3x^2 + 6x = 0$

$3x(-x + 2) = 0$

$\underline{x = 0}$, $\boxed{x = 2}$

$-8 + 12 - 2 = 2$

$f''(x) = -6x + 6$

$f''(0) = 6 > 0$ concave up ∪

rel min at $x = 0$, $y = -2$

$f''(2) = -12 + 6 = -6 < 0$

Concave down ∩

Rel MAX

(2, 2)

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12. Finding maxima and minima: a) Calculate critical points ($y' = 0$ or undefined) b) Determine concavity at critical points. $y'' > 0$ (concave up ∪ relative min), $y'' < 0$ (concave down ∩ relative max) c) Check y value at any given end points if y is defined on an interval [a, b].

$f(x) = x^3 - 3x + 5; [-1, 3]$

Absolute maximum: 23, absolute minimum: 3

$f'(x) = 3x^2 - 3 = 0$

$3(x^2 - 1) = 0$

$x = -1 \quad x = +1$

check $\quad f(-1) = -1 + 3 + 5 = 7$

$f(1) = 1 - 3 + 5 = 3$ MIN abs

check end points (-1) checked $f(-1) = 7$

check $x = 3 \quad f(3) = 27 - 9 + 5 = 23$ MAX

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13. Fundamental Theorem of Calculus Part 1. If $F(x) = \int_a^x f(t)\,dt$. *Then* $F'(x) = f(x)$

However, if $F(x) = \int_a^{g(x)} f(t)\,dt$. *Then* $F'(x) = f(x)\ g'(x)$ *Remember the Chain Rule!*

$$\frac{d}{dx}\int_0^x 7t\cos(t^6)\,dt = 7x\cos(x^6)$$ OK

$$\frac{d}{dx}\int_0^{\sin x} \frac{1}{1-t^2}\,dt = \frac{\cos x}{1-\sin^2 x}$$ OK

14. Fundamental Theorem of Calculus Part 2. If $F(x) is\ an\ antiderivative\ of\ f(x) on\ [a, b].\ then$

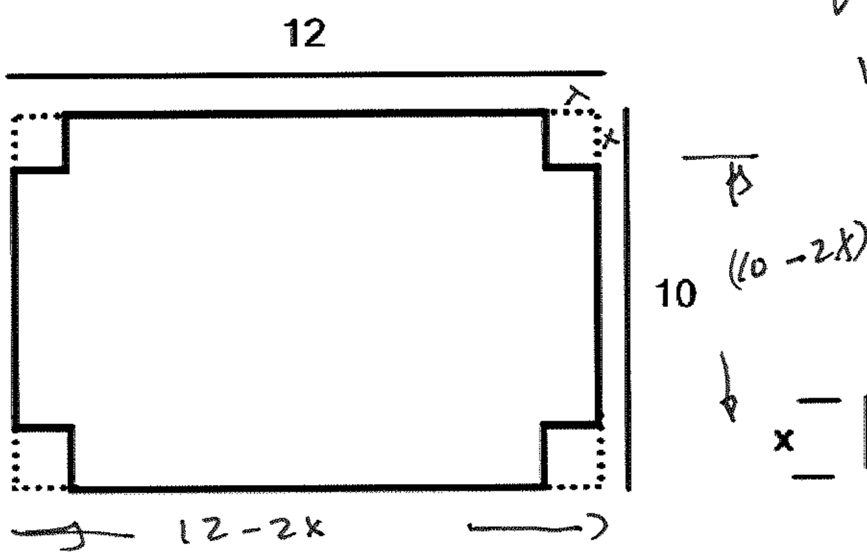
$$\int_a^b f(x)\,dx = F(b) - F(b)$$

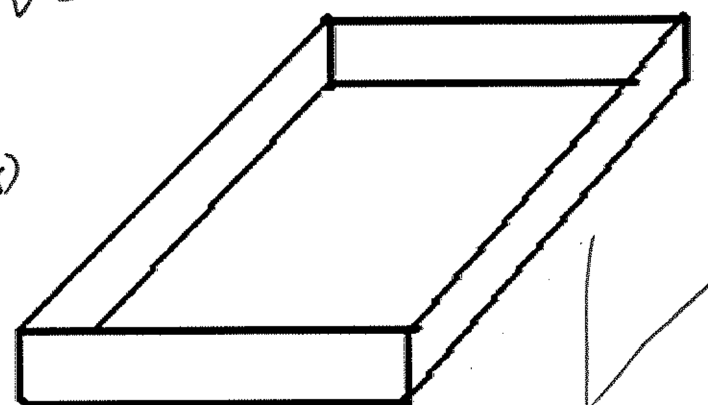
15. Optimization

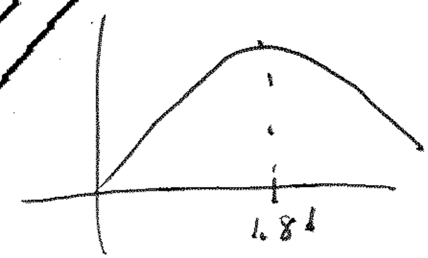
A sheet of metal 12 inches by 10 inches is to be used to make a open box. Squares of equal sides x are cut out of each corner then the sides are folded to make the box. Find the value of x that makes the volume maximum. What are the dimensions and the volume?

$V = x(10-2x)(12-2x)$

$V = x(120 - 24x - 20x + 4x^2)$







V(1.81) = 96.77 sq. inches. Dimensions 8.38" x 6.38" x 1.81"

This problem can be solved with Calculus or by graphing the correct function.

$V = x(120 - 44x + 4x^2) = 120x - 44x^2 + 4x^3$

$V' = 120 - 88x + 12x^2 = 0$

$= 4(30 - 22x + 3x^2) = 0$

$= \dfrac{22 \pm \sqrt{(22)^2 - 4(3)(30)}}{6}$

$\begin{cases} x = 1.81 \\ x = 5.52 \end{cases}$ X reject

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16. Related rates.

$\frac{dA}{dt} = 9$

1) Water is falling on a surface, wetting a circular area that is expanding at a rate of 9 mm^2/s. How fast is the radius of the wetted area expanding when the radius is 166 mm? (Round your answer to four decimal places.)

0.0086 mm/s

2) Water is discharged from a pipeline at a velocity v (in ft/sec) given by v = 1064p(1/2), where p is the pressure (in psi). If the water pressure is changing at a rate of 0.266 psi/sec, find the acceleration (dv/dt) of the water when p = 32.0 psi.

25.0 ft/sec^2

$A = \pi r^2$

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

$9 = 2\pi(166) \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{9}{2\pi(166)} = .0086$ mm/sec

$V = 1064\, p^{1/2}$

$\frac{dv}{dt} = \frac{1}{2}(1064)\, p^{-1/2} \frac{dp}{dt} = \frac{532}{\sqrt{p}} \frac{dp}{dt}$

at $p = 32$

$\frac{dv}{dt} = \frac{532}{\sqrt{32}}(.266) = 25.01$

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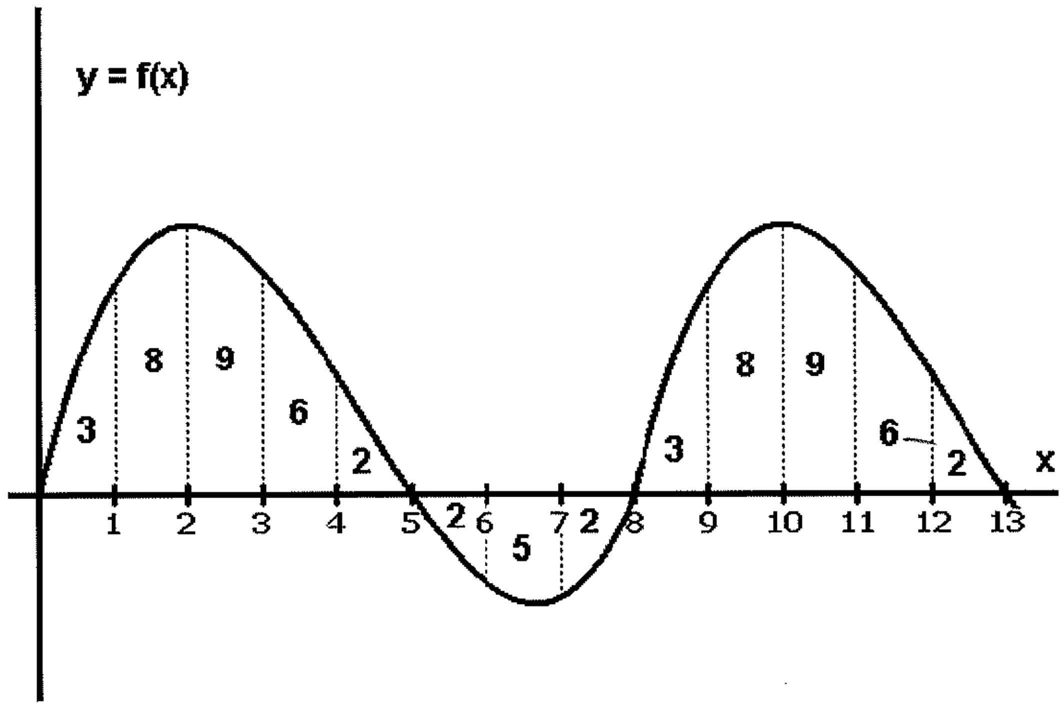
Additional Note on Fundamental Theorem of Calculus Part 1.



t	S(t)
0	0
1	3
2	11
3	20
4	26
5	28
6	26
7	21
8	19
9	22
10	30
11	39
12	45
13	47

Given $f(x)$ $in\ the\ graph\ above\ we\ define\ s(t) = \int_0^t f(x)dx.$

The numbers under the curve represent the true area in each section. On the chart below sketch s(t) and then answer the questions below.

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Given $f(x)$ $in\ the\ graph\ above\ we\ define\ s(t) = \int_0^t f(x)dx.$

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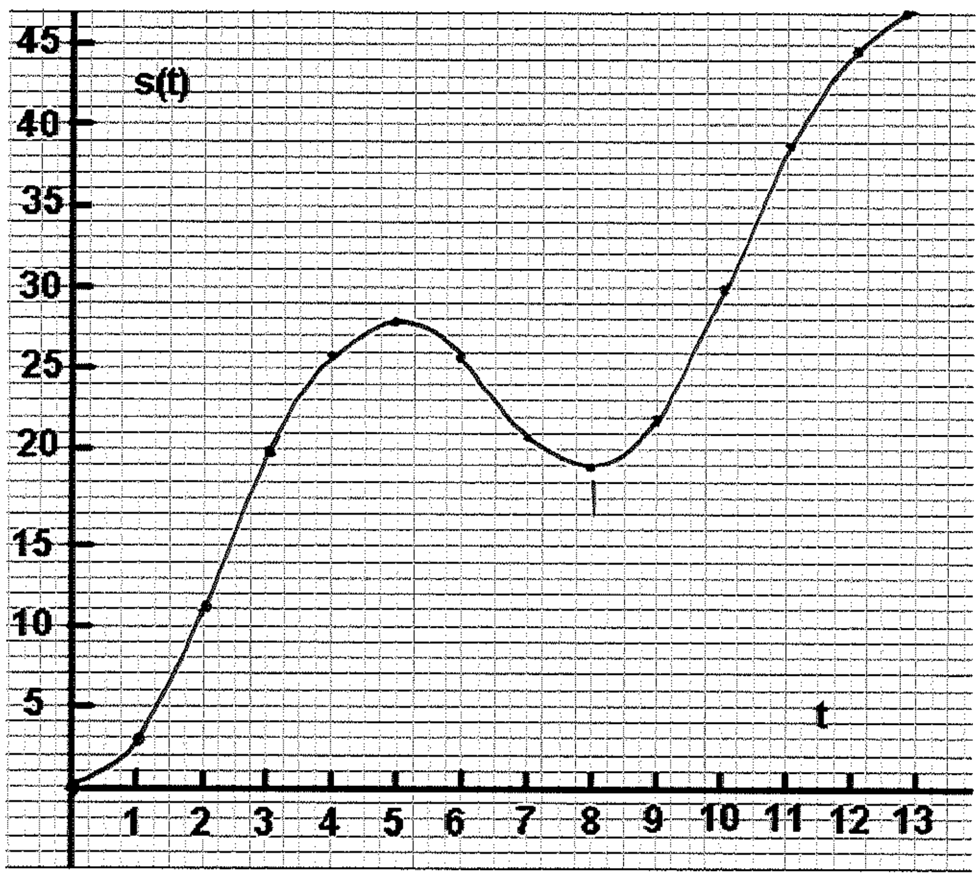

What is s'(t) ? f(t)

What is s''(t) ? f'(t)

At what value of t does s(t) reach its first relative max? 5 . What is the sign of s''(t) at that point? Concave down −

At what value of t does s(t) reach its first relative min? 8 . What is the sign of s''(t) at that point? +