## Quantitative Skills \& Reasoning - Math 1001

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## Logic



## George Boole

Logic is, basically, the study of valid reasoning. When searching the internet, we use Boolean logic - terms like "and" and "or" - to help us find specific web pages that fit in the sets we are interested in. After exploring this form of logic, we will look at logical arguments and how we can determine the validity of a claim.

## Boolean Logic

Boolean logic combines multiple statements that are either true or false into an expression that is either true or false.

In connection to sets, a search is true if the element is part of the set.


AND
Using AND, this search would only retrieve results with Peanut Butter and Jelly.


OR
Using OR, this search would retrieve results with peanut butter, with jelly, and with both.


NOT Jelly
Using NOT, this search would retrieve results with peanut butter, and exclude those with jelly or PB with jelly.



## Connection to Set Operations

- $A$ and $B \quad$ elements in the intersection $A \cap B$
- $A$ or $B \quad$ elements in the union $A \cup B$
- not $A \quad$ elements in the complement $A^{c}$
- Notice here that or is not exclusive. This is a difference between the Boolean logic use of the word and common everyday use. When your significant other asks "do you want to go to the park or the movies?" they usually are proposing an exclusive choice - one option or the other, but not both. In Boolean logic, the or is not exclusive - more like being asked at a restaurant "would you like fries or a drink with that?" Answering "both, please" is an acceptable answer.


## Example 1

Suppose we are searching a library database for Mexican universities. Express a reasonable search using Boolean logic.

We could start with the search "Mexico and university", but would be likely to find results for the U.S. state New Mexico. To account for this, we could revise our search to read:
Mexico and university not "New Mexico"

## Example 2

Describe the numbers that meet the condition: even and less than 10 and greater than 0

The numbers that satisfy all three requirements are $\{2,4,6,8\}$

Sometimes statements made in English can be ambiguous. For this reason, Boolean logic uses parentheses to show precedent, just like in algebraic order of operations.

## Example 3

Describe the numbers that meet the condition:
odd number and less than 20 and greater than 0 and (multiple of or multiple of 5)

The first three conditions limit us to the set
$\{1,3,5,7,9,11,13,15,17,19\}$

The last grouped conditions tell us to find elements of this set that are also either a multiple of 3 or a multiple of 5 .

This leaves us with the set $\{3,5,9,15\}$

## Statements and Conditionals

- A statement is either true or false.
- A conditional is a compound statement of the form "if $p$ then $q$ " or "if $p$ then $q$, else $s$ ".

A statement is something that is either true or false. A statement like $3<5$ is true; a statement like "a rat is a fish" is false. A statement like " $x<5$ " is true for some values of $x$ and false for others. When an action is taken or not depending on the value of a statement, it forms a conditional.

## Example 4

In common language, an example of a conditional statement would be "If it is raining, then we'll go to the mall. Otherwise we'll go for a hike."

The statement "If it is raining" is the condition - this may be true or false for any given day. If the condition is true, then we will follow the first course of action, and go to the mall. If the condition is false, then we will use the alternative, and go for a hike.

## Quantified Statements

Words that describe an entire set, such as "all", "every", or "none", are called universal quantifiers because that set could be considered a universal set. In contrast, words or phrases such as "some", "one", or "at least one" are called existential quantifiers because they describe the existence of at least one element in a set.

## Quantifiers

A universal quantifier states that an entire set of things share a characteristic.

An existential quantifier states that a set contains at least one element.

## Negating a Quantified Statement

- The negation of "all $A$ are $B$ " is "at least one $A$ is not $B$ ".

All Those In Buckhead are millionaires. (I know John there who is not)

- The negation of "no $A$ are $B$ " is "at least one $A$ is $B$ ".

Nobody in my graduating class is an engineer. (I know Sally is an engineer)

- The negation of "at least one $A$ is $B$ " is "no $A$ are $B$ ".

We have at least one singer in our class. (nobody in our class sings)

- The negation of "at least one $A$ is not $B$ " is "all $A$ are $B$ ".

A least one of my children did not have measles. (All my children had measles)

## Example 5

"Somebody brought a flashlight." Write the negation of this statement.

The negation is "Nobody brought a flashlight."

## Example 6

"There are no prime numbers that are even." Write the negation of this statement.

The negation is "At least one prime number is even."

## Conjunction (and), Disjunction (or), Negation (not)

## Symbols

The symbol $\wedge$ is used for and: $A$ and $B$ is notated $A \wedge B$
The symbol $V$ is used for or: $\quad A$ or $B$ is notated $A \vee B$
The symbol $\sim$ is used for not: $\quad \operatorname{not} A$ is notated $\sim A$

## Example 7

Translate each statement into symbolic notation. Let $P$ represent "I like Pepsi" and let C represent "I like Coke".
a. I like Pepsi or I like Coke.
b. I like Pepsi and I like Coke.
c. I do not like Pepsi.
d. It is not the case that I like Pepsi or Coke.
e. I like Pepsi and I do not like Coke.

## Example 7

Translate each statement into symbolic notation. Let $P$ represent "I like Pepsi" and let C represent "I like Coke".
a. I like Pepsi or I like Coke. P V C
b. I like Pepsi and I like Coke. $P \wedge C$
c. I do not like Pepsi. $\sim P$
d. It is not the case that I like Pepsi or Coke. $\sim(P \vee C)$
e. I like Pepsi and I do not like Coke. $P \wedge \sim C$

## Truth Table

Because complex Boolean statements can get tricky to think about, we can create a truth table to keep track of what truth values for the simple statements make the complex statement true and false.

## Truth table

A table showing what the resulting truth value of a complex statement is for all the possible truth values for the simple statements.

## Example 8

Suppose you're picking out a new couch, and your significant other says "get a sectional or something with a chaise".

This is a complex statement made of two simpler conditions: "is a sectional", and "has a chaise". For simplicity, let's use $S$ to designate "is a sectional", and $C$ to designate "has a chaise".

A truth table would look like:

| S | C | S or C |
| :---: | :---: | :---: |
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

In the table, T is used for true, and F for false. In the first row, if $S$ is true and $C$ is also true, then the complex statement " $S$ or $C$ " is true. This would be a sectional that also has a chaise, which meets our desire. (Remember that or in logic is not exclusive; if the couch has both features, it meets the condition.)

## Basic Truth Tables

| Conjunction |  |  |
| :---: | :---: | :---: |
| A | B | A $\wedge$ B |
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |


| Disjunction |  |  |
| :---: | :---: | :---: |
| A | B | A V B |
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |


$A \wedge B$ is true only if $A$ is true and $B$ is true
$A \bigvee B$ is true if $A$ is true, or if $B$ is true, or if both $A$ and $B$ are true

## Example 9

Create a truth table for the statement $A \vee \sim B$
When we create the truth table, we need to list all the possible truth value combinations for $A$ and $B$. Notice how the first column contains 2 Ts followed by 2 Fs , and the second column alternates T, F, T, F. This pattern ensures that all 4 combinations are considered.


## Example 9 (cont)

After creating columns with those initial values, we create a third column for the expression $\sim B$. Now we will temporarily ignore the column for $A$ and write the truth values for $\sim B$.

| A | B | $\sim$ B |
| :---: | :---: | :---: |
| T | T | F |
| T | F | T |
| F | T | F |
| F | F | T |

## Example 9 (cont)

Next we can find the truth values of $A \vee \sim B$, using the first and third columns.

| A | B | $\sim$ B | A $V \sim$ B |
| :---: | :---: | :---: | :---: |
| T | T | F | T |
| T | F | T | T |
| F | T | F | F |
| F | F | T | T |

## Conditional

A conditional is a logical compound statement in which a statement $p$, called the antecedent, implies a statement $q$, called the consequent.

A conditional is written as $\mathbf{p} \rightarrow \mathbf{q}$ and is translated as "if $\mathbf{p}$, then $\mathbf{q}$ ".
Consequent - a thing that follows another. (I say the consequence)

A conditional asserts that if its antecedent (p) is true, its consequent ( $q$ ) is also true; any conditional with a true antecedent and a false consequent must be false.

The English statement "If it is raining, then there are clouds is the sky" is a conditional statement. It makes sense because if the antecedent ( $p$ ) "it is raining" is true, then the consequent (q) "there are clouds in the sky" must also be true. $\mathbf{p} \rightarrow \mathbf{q}$

Notice that the statement tells us nothing of what to expect if it is not raining; there might be clouds in the sky, or there might not. If the antecedent is false, then the consquent becomes irrelevant.

In traditional logic, a conditional is considered true as long as there are no cases in which the antecedent is true and the consequent is false.

## Truth table for the conditional

| $p$ | $q$ | $p \rightarrow q$ |
| :---: | :---: | :---: |
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

It is raining and there are clouds in the sky -True
It is raining and there are no clouds in the sky -False
It is not raining and there are clouds in the sky -True if can't prove false It is not raining and no clouds in the sky -True if can't prove false

Again, if the antecedent $p$ is false, we cannot prove that the statement is a lie, so the result of the third and fourth rows is true. "You are innocent until proved guilty."

For any conditional, there are three related statements, the converse, the inverse, and the contrapositive.

## Related Statements

The original conditional is "if $p$, then $q$ " $q \rightarrow p$ true
The converse is "if $q$, then $p$ " $\quad q \rightarrow p$ may/may not be true
The inverse is "if not $p$, then not $q$ " $\quad \sim p \rightarrow \sim q$ may/may not be true
The contrapositive is "if not $q$, then not $p$ " $\sim q \rightarrow \sim p$ always true

Consider again the conditional "If it is raining, then there are clouds in the sky." It seems reasonable to assume that this is true. $\mathrm{p} \rightarrow \mathrm{q}$

The converse would be "If there are clouds in the sky, then it is raining." This is not always true. $\quad q \rightarrow p$

The inverse would be "If it is not raining, then there are not clouds in the sky." Likewise, this is not always true. ${ }^{\sim} \mathrm{p} \rightarrow{ }^{\sim} \mathrm{q}$

The contrapositive would be "If there are not clouds in the sky, then it is not raining." This statement is true, and is equivalent to the original conditional. $\quad \sim \mathrm{q} \rightarrow \sim_{p}$

Looking at truth tables, we can see that the original conditional and the contrapositive are logically equivalent, and that the converse and inverse are logically equivalent.

|  |  | Conditional | Converse | Inverse | Contrapositive |
| :---: | :---: | :---: | :---: | :---: | :---: |
| p | q | $\mathrm{p} \rightarrow \mathrm{q}$ | $\mathrm{q} \rightarrow \mathrm{p}$ | $\sim \mathrm{p} \rightarrow{ }^{\sim} \mathrm{q}$ | $\sim \mathrm{q} \rightarrow{ }^{\sim} \mathrm{p}$ |
| T | T | T | T | T | T |
| T | F | F | T | T | F |
| F | T | T | F | F | T |
| F | F | T | T | T | T |

Guide for Conditional

| $p$ | $q$ | $p \rightarrow q$ |
| :---: | :---: | :---: |
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

Equivalence A conditional statement and its contrapositive are logically equivalent. The converse and inverse of a conditional statement are logically equivalent.


In other words, the original statement and the contrapositive must agree with each other; they must both be true, or they must both be false. Similarly, the converse and the inverse must agree with each other; they must both be true, or they must both be false.

## Example 11

Suppose this statement is true: "If I eat this giant cookie (p), then I will feel sick $\mathbf{q}$." $\mathbf{p} \rightarrow \mathbf{q}$ Which of the following statements must also be true?
a. If I feel sick, then I ate that giant cookie. -This is the converse $\mathbf{q} \rightarrow \mathbf{p}$ which may or may not be true - I Could be sick for other reasons
b. If I don't eat this giant cookie, then I won't feel sick. This is the inverse $\sim p \rightarrow \sim q$, again this may not be true. Could be sick for some other reason.
c. If I don't feel sick, then I didn't eat that giant cookie. This is the contrapositive ${ }^{\sim} q \rightarrow \sim \mathcal{p}$ This is true because if I didn't feet sick then I must not have eaten the giant cookie.

Notice again that the original statement and the contrapositive have the same truth value (both are true), and the converse and the inverse have the same truth value (both are false).

## Equivalence

A conditional statement and its contrapositive are logically equivalent.

The converse and inverse of a conditional statement are logically equivalent.

The Negation of a Conditional
The negation of a conditional statement is logically equivalent to a conjunction of the antecedent and the negation of the consequent.
$\sim(p \rightarrow q)$ is equivalent to $p \wedge \sim q$

| p | q | $\mathrm{p} \rightarrow \mathrm{q}$ | $\sim(\mathrm{p} \rightarrow \mathrm{q})$ | $\mathrm{p} \wedge \sim \mathrm{q}$ |
| :---: | :---: | :---: | :---: | :---: |
| T | T | T | F | F |
| T | F | F | T | T |
| F | T | T | F | F |
| F | F | T | F | F |

p AND NOT q

Which of the following statements is equivalent to the negation of "If you don't grease the pan, then the food will stick to it" ?
a. I didn't grease the pan and the food didn't stick to it. b. I didn't grease the pan and the food stuck to it. c. I greased the pan and the food didn't stick to it.
a. This is correct; it is the conjunction of the antecedent and the negation of the consequent. To disprove that not greasing the pan will cause the food to stick, I have to not grease the pan and have the food not stick.
b. This is essentially the original statement with no negation; the "if...then" has been replaced by "and".
c. This essentially agrees with the original statement and cannot disprove it.

## Biconditional

A biconditional is a logical conditional statement in which the antecedent and consequent are interchangeable.

A biconditional is written as $p \leftrightarrow q$ and is translated as " $p$ if and only if $q$ ".

## Truth table for the biconditional

| $p$ | $q$ | $p \leftrightarrow q$ |
| :---: | :---: | :---: |
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

A biconditional is considered true as long as the antecedent and the consequent have the same truth value; that is, they are either both true or both false.

Suppose this statement is true: "The garbage truck comes down my street if and only if it is Thursday morning." Which of the following statements could be true?
a. It is noon on Thursday and the garbage truck did not come down my street this morning. b. It is Monday and the garbage truck is coming down my street. c. It is Wednesday at 11:59PM and the garbage truck did not come down my street today.
a. This cannot be true. This is like the second row of the truth table; it is true that I just experienced Thursday morning, but it is false that the garbage truck came.
b. This cannot be true. This is like the third row of the truth table; it is false that it is Thursday, but it is true that the garbage truck came.
c. This could be true. This is like the fourth row of the truth table; it is false that it is Thursday, but it is also false that the garbage truck came, so everything worked out like it should.

